Elasticity

Department of Civil Engineering, National Cheng Kung University

***Return question-paper after exam**
***Open-book exam**

---

1. **(30 points)** Given the relations

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}, \quad J_2 = \frac{1}{2}s_{ij}s_{ji}, \quad J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki},$$

where σ and s are symmetric second-order tensors. Show that

(a) $s_{ii} = 0$.

(b) $J_{2,\sigma ij} = s_{ij}$.

(c) $J_{3,\sigma ij} = s_{ik}s_{kj} - 2J_2\delta_{ij}/3$.

2. **(30 points)** For isotropic linear elastic materials, prove the following relations between the elastic moduli E (Young's modulus), G (shear modulus), ν (Poisson's ratio) and k (bulk modulus):

$$\nu = \frac{3k - E}{6k}, \quad k = \frac{GE}{9G - 3E}.$$

Using the fact that for an isotropic body the principal axes of stress and strain coincide and assuming that the stress-strain relation is linear so that superposition holds, derive

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{mm}\delta_{ij},$$

directly from the definitions of E and ν given by

$$E = \frac{\sigma_{11}}{\varepsilon_{11}}, \quad \nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}},$$

in simple tension test.

3. **(30 points)** The elastic complementary energy density U^c is defined viz.

$$U^c \equiv \int_0^{\sigma_{ij}} \varepsilon_{ij}\, d\sigma_{ij}.$$

Based on this concept, the behavior of an isotropic nonlinear elastic material is described by the following assumed polynomial expression for U^c:

$$U^c\left(I_1, J_2, J_3\right) = aI_1^2 + bJ_2 + cJ_3^2,$$

where a, b and c are constants and I_1, J_2 and J_3 are the stress invariants defined in Problem

1 of this exam.

(a) Derive the stress-strain relations for this material in terms of the constants a, b and c.

(b) Derive the stress-strain relations in simple tension. Find an expression for the tangent Young's modulus E_t defined viz.

$$E_t \equiv \frac{d\sigma}{d\varepsilon},$$

in simple tension, in terms of the stress σ. What is the value of the initial tangent Young's modulus $E_t(0)$ (at $\sigma = 0$)?

(c) Show that the constitutive equations are reduced to those of the isotropic linear elastic material for $c = 0$. Find the relations between the constants a, b, and the elastic moduli E, ν for this case.

4. **(10 points)** For isotropic materials, show that the principal axes of strain coincide with the principal axes of stress. Further, show that the principal stresses can be expressed in terms of the principal strains via

$$\sigma_i = 2\mu\varepsilon_i + \lambda\varepsilon_{kk},$$

where σ_i and ε_i are the principal stresses and strains, respectively.

NCKU 2019 Fall Semester

Qualifying Examination for Ph. D. Candidates

Department: Civil Engineering
Course: Dynamics of Structure
Qualifying score: 60
Time: 100 minutes

1. Develop the equation governing the rotational motion of the system pivoted at point A in Figure 1. The rigid rod is massless. (20%)

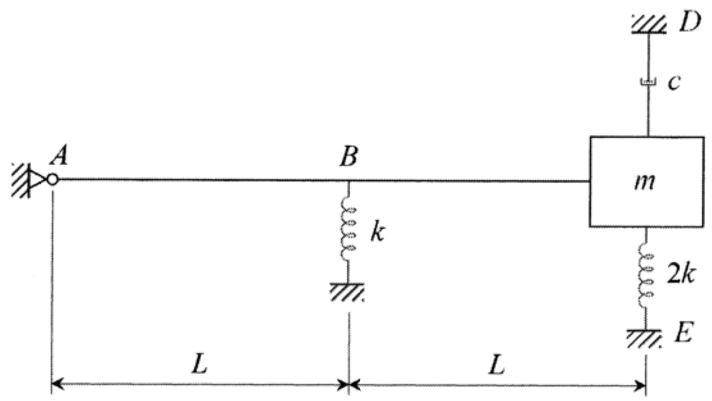


Figure 1

2. One story shear building shown in Fig. 2 is subjected to the ground shaking. The section of each column is 10 cm by 10 cm. The unit weight of the floor is 2000 N/m. The input ground acceleration is 200 sin 5t (cm/sec^2). It is assumed the damping ratio of the system is 5% and the modulus of elasticity E for the column is 2×10^5 N/cm^2. Please determine the maximum relative displacement on the top of the column, and the maximum shear force for column A and column B. (20%)

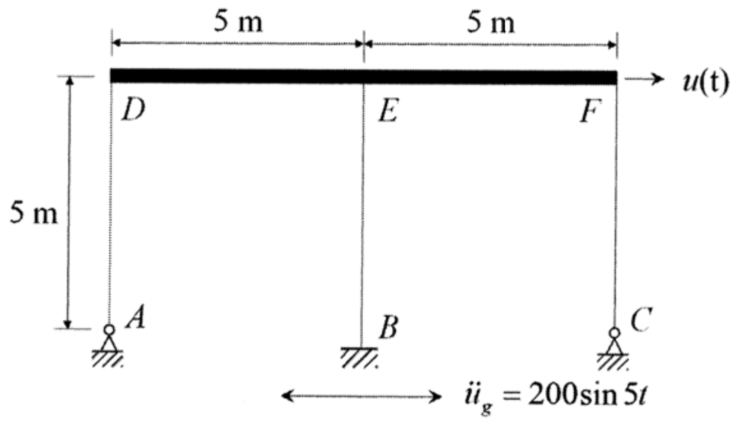


Figure 2

3. Short questions about an underdamped SDOF system:

(1) Explain the meaning of phase angle. (10%)

(2) How to design an instrument to measure the ground acceleration ? (10%)

4. For the two-story shear building of Fig. 3 excited by horizontal ground motion $\ddot{u}_g(t)$, please determine:

(a) the natural frequencies (10%)

(b) mode shapes (10%)

(c) the floor displacement responses in terms of $D_n(t)$ (10%)

(d) the story shear responses in terms of $A_n(t)$ (10%)

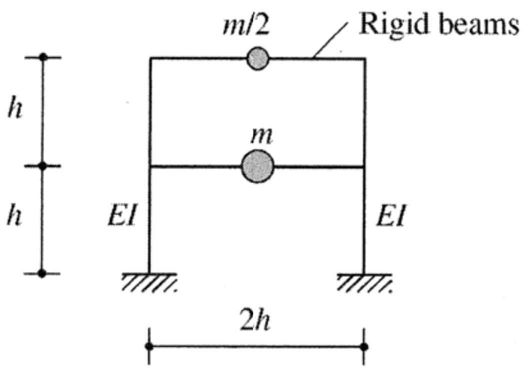


Figure 3

Qualification Examination (108.10)
Finite Element Method

(1) Bending of a beam is governed by the fourth-order equation

$$EI\frac{d^4w}{dx^4}=q(x) \qquad \text{in the region } 0\le x\le L, \tag{1-1}$$

where E is the modulus of elasticity, I is the moment of inertia of the beam cross-sectional area, and $q(x)$ is the transverse loading per unit length. The potential energy functional of a typical beam element is given as

$$I^{(e)}(w^{(e)})=\int_{x_1^{(e)}}^{x_2^{(e)}}\left[\frac{EI}{2}\left(\frac{d^2w^{(e)}}{dx^2}\right)^2-qw^{(e)}\right]dx-Q_1^{(e)}w_1^{(e)}-Q_2^{(e)}\left(\frac{dw^{(e)}}{dx}\right)_1-Q_3^{(e)}w_2^{(e)}-Q_4^{(e)}\left(\frac{dw^{(e)}}{dx}\right)_2, \tag{1-2}$$

where the coordinates are shown in Fig. 1a; $x_1^{(e)}$ and $x_2^{(e)}$ denote the global coordinates of nodes 1 and 2 of a typical element, respevtively; $Q_i^{(e)}$ $(i=1-4)$ denote the nodal forces of a typical beam element and are shown in Fig. 1b; and the rotations $\theta_i^{(e)}$ (i=1 and 2) at each node are defined as $\theta_i^{(e)}=\left(\frac{dw}{dx}\right)_i^{(e)}$.

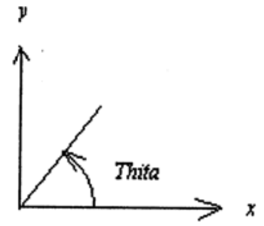


Fig.1a

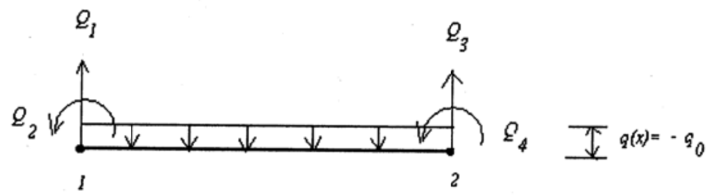


Fig. 1b

(a) Derive the weak form of this problem using the variational approach with Eq. (1-2), and discuss the selection of an interpolation function for a beam element and identify the nodal degrees of freedom.

(10%)

(b) For a beam element, we assume that the displacement interpolation is

$$w^{(e)}(\bar{x})=w_1^{(e)}N_1(\bar{x})+\left(\frac{dw}{dx}\right)_1^{(e)}N_2(\bar{x})+w_2^{(e)}N_3(\bar{x})+\left(\frac{dw}{dx}\right)_2^{(e)}N_4(\bar{x}), \tag{1-3}$$

in which $N_i(\bar{x})$ $(i=1-4)$ denote the shape functions (or interpolation functions).

Describe how do you derive the shape functions ($N_i(\bar{x})$), which are expressed in terms of the local coordinates $\bar{x}$, and given as

$$N_1(\bar{x})=1-3\left(\frac{\bar{x}}{h_e}\right)^2+2\left(\frac{\bar{x}}{h_e}\right)^3, \qquad N_2(\bar{x})=\bar{x}\left(\frac{\bar{x}}{h_e}-1\right)^2,$$

$$N_3(\bar{x})=\left(\frac{\bar{x}}{h_e}\right)^2\left(3-2\frac{\bar{x}}{h_e}\right), \qquad N_4(\bar{x})=\frac{\bar{x}^2}{h_e}\left(\frac{\bar{x}}{h_e}-1\right),$$

(10%)

where h_e denotes the length of a typical beam element.

(c) Use the weak formulation, that you obtained in Problem 1a, to determine the beam element stiffness matrix $\left[\mathbf{K}^{(e)}\right]$ and forcing vector $\left\{\mathbf{F}^{(e)}\right\}$. (refer Eq. (2-1)) (10%)

(2) Consider a loaded Euler-Bernoulli beam structure as shown in Fig. 2a. Analyze this beam using two-beam-element model with uniform spacing. In particular, this analysis should give (a) The assembled stiffness matrix and force vector, (b) The specified global displacements and forces, and the equilibrium conditions, (c) The condensed matrix equations for the primary unknowns separately, (d) the solutions of unknown nodal displacements and rotations, and the reactions. (30%)
Noted that the numberings of nodal variables and beam elements are shown in Fig. 2b, and the element stiffness matrix $\left[\mathbf{K}^{(e)}\right]$ and forcing vector $\left\{\mathbf{F}^{(e)}\right\}$ are given as

$$\left[\mathbf{K}^{(e)}\right]=\left(\frac{EI}{h_e^3}\right)\begin{bmatrix} 12 & 6h_e & -12 & 6h_e \\ 6h_e & 4h_e^2 & -6h_e & 2h_e^2 \\ -12 & -6h_e & 12 & -6h_e \\ 6h_e & 2h_e^2 & -6h_e & 4h_e^2 \end{bmatrix}, \qquad \left\{\mathbf{F}^{(e)}\right\}=\begin{Bmatrix} -q_0 h_e/2 \\ -q_0 h_e^2/12 \\ -q_0 h_e/2 \\ +q_0 h_e^2/12 \end{Bmatrix}+\begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \\ Q_3^{(e)} \\ Q_4^{(e)} \end{Bmatrix} \tag{2-1}$$

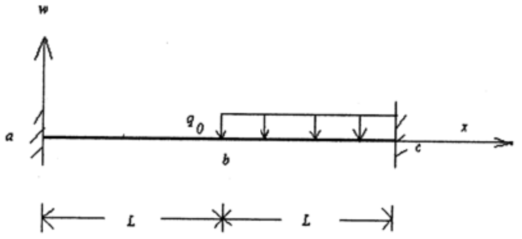


Fig. 2a

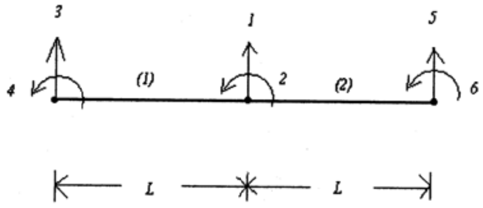


Fig. 2b

(3) If the nodal values of the element shown in Fig. 3 are $u_i=\hat{u}_i\ (i=1,2,3)$, compute u, $\partial u/\partial x$ and $\partial u/\partial y$ at point $(x, y)=(0.25, 0.25)$. (20%)

Hint: The linear interpolation functions $\phi_i^{(e)}(x,y)=\frac{1}{2A_e}\left(\alpha_i^{(e)}+\beta_i^{(e)}x+\gamma_i^{(e)}y\right)\ (i=1,2,3)$ and $\alpha_i^{(e)}=x_j y_k-x_k y_j$, $\beta_i^{(e)}=y_j-y_k$, $\gamma_i=-(x_j-x_k)$, ($i\neq j\neq k$; i, j and k permute in a natural order).

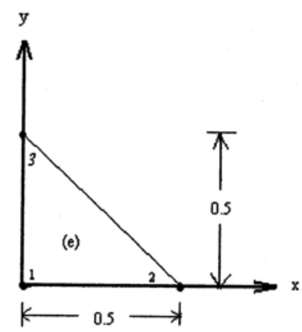


Fig. 3

(4) Describe how to determine the shape functions for the five-node rectanguler element. (20%)

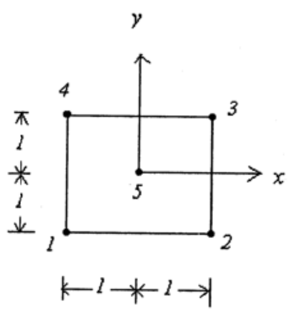


Fig. 4

108學年度第一學期博士學位候選人資格考試

工程統計

作答方式：Open Book　考試時間：100分鐘　及格分數：60分

1. (a) What are the major differences between **binomial** and **Poisson** distributions? (10%)
   (b) What are the major differences between **normal** and **lognormal** distributions? (10%)
   (c) What are the major differences between **estimation** and **test**? (10%)

2. Given 10 test results as follows:
   2216, 2225, 2313, 2237, 2301, 2255, 2281, 2275, 2207, 2290.
   (a) Determine the probability of a random result being **greater than** 2300 if the data is **normally** distributed. (10%)
   (b) Construct a **95% confidence interval** on the **mean** if the data is **normally** distributed. (10%)
   (c) Construct a **95% upper confidence limit** on the **variance** if the data is **normally** distributed. (10%)
   (d) The null and alternative hypotheses are

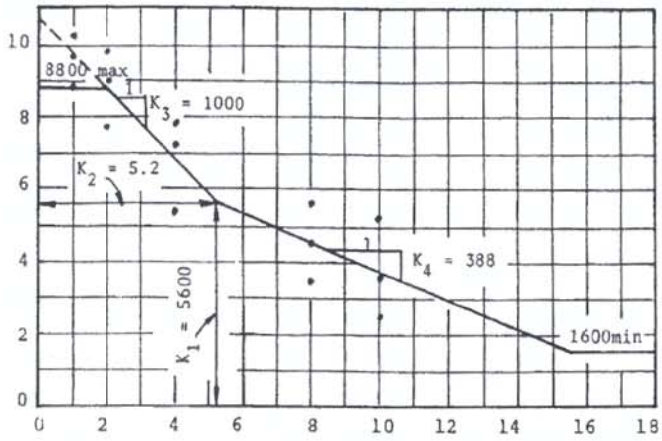
$$\begin{cases} H_0 : \mu = 2300 \\ H_1 : \mu \neq 2300 \end{cases}.$$

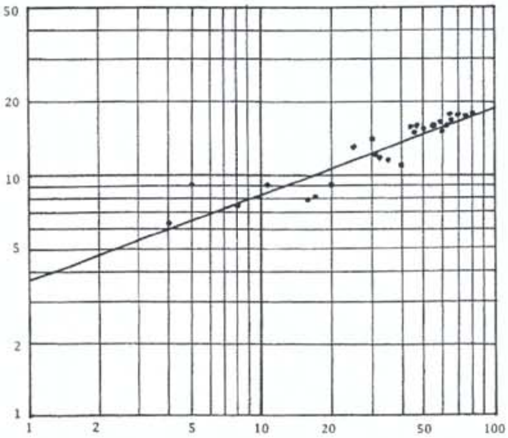
   Test the hypothesis at the **5% significance level**. (10%)

3. Given n sample pairs, (x_i, y_i), $i = 1, 2, \ldots, n$, to perform two linear regression analyses, $y = ax + b$ and $x = \alpha y + \beta$. Are the results the same line? Give two major reasons, without any mathematical calculation, to support your answer. (18%)

4. What is the purpose on estimating the coefficient of determination, R^2? Under what conditions could the coefficient of determination be related to the sample correlation coefficient, $\hat{\rho}$? (12%)

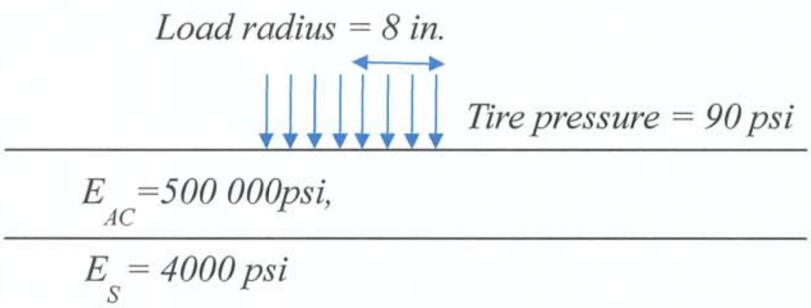
1. Please add the labels of x-axis and y-axis for the figures below. (10%)





2. Use Westergaard solution to analyze a concrete slab (150in×210in×14in) resting on Winkler foundation (k=100 psi/in) and loaded with a single wheel.
   (a) calculate the maximum tensile stress in the concrete slab. **(5%)**
   (b) calculate the maximum deflection in the concrete slab **(5%)**
   (c) calculate the maximum compressive stress in the subgrade. **(5%)**
   (d) calculate the shrinkage length under the conditions $\Delta T=60°F$, $\alpha=5.5\times10^{-6}/°F$, $\varepsilon=1\times10^{-4}$, $C=0.65$. **(5%)**

3. A circular load with a radius of 8 inches and a uniform pressure of 90 psi is applied on a two-layer system. The subgrade has an elastic modulus of 4000 psi and can support a maximum vertical stress of 8 psi. If an HMA has an elastic modulus of 500000 psi, what is the required thickness of a full-depth pavement? **(20%)**



4. According to AASHO Road Test, a rigid pavement with D=4 in. subjected a tandem-axle load of 32 kips, based on a p_t=2.5
   (a) What is the damage compared with that of a 18 kips single axle load ? **(10%)**
   (b) What is the tandem-axle load on a flexible pavement with SN=5 that is equivalent to that on rigid pavement? **(10%)**

5. Explain the differences between Young's modulus and resilient modulus. (10%)

6. Please briefly introduce falling weight deflectometer. (10%)
   and the application of the measured data. **(10%)**